4-5 Determinants

Main Ideas

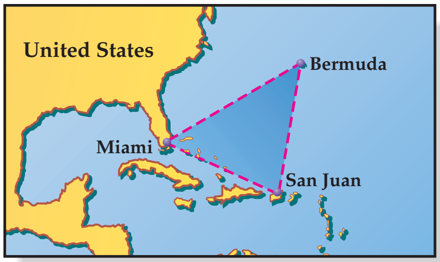
- Evaluate the determinant of a 2×2 matrix.
- Evaluate the determinant of a 3×3 matrix.

New Vocabulary

determinant
second-order determinant
third-order determinant
expansion by minors
minor

GET READY for the Lesson

The "Bermuda Triangle" is an area located off the southeastern Atlantic coast of the United States that is noted for a high incidence of unexplained losses of ships, small boats, and aircraft. Using the coordinates of the vertices of this triangle, you can find the value of a determinant to approximate the area of the triangle.



Determinants of 2 × 2 Matrices Every square matrix has a number associated with it called its **determinant.** The determinant of $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$ can be represented by $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}$ or $\det\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$. The determinant of a 2×2 matrix is called a **second-order determinant.**

KEY CONCEPT — *Second-Order Determinant*

Words The value of a second-order determinant is found by calculating the difference of the products of the two diagonals.

Symbols $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Example $\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix} = 3(5) - (-1)(2) = 17$

EXAMPLE 1 Second-Order Determinant

Find the value of the determinant $\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix}$.

$\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} = (-2)(8) - 5(6)$ Definition of determinant

$= -16 - 30$ or -46 Multiply.

CHECK Your Progress

Find the value of each determinant.

1A. $\begin{vmatrix} 7 & 4 \\ -3 & 2 \end{vmatrix}$

1B. $\begin{vmatrix} -4 & 6 \\ -3 & -2 \end{vmatrix}$

Study Tip

Determinants

Note that only square matrices have determinants.

Determinants of 3 × 3 Matrices

Determinants of 3 × 3 matrices are called **third-order determinants**. One method of evaluating third-order determinants is **expansion by minors**. The **minor** of an element is the determinant formed when the row and column containing that element are deleted.

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{vmatrix} \quad \text{The minor of } a \text{ is } \begin{vmatrix} e & f \\ h & i \end{vmatrix}.$$

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & \cancel{e} & f \\ g & \cancel{h} & i \end{vmatrix} \quad \text{The minor of } b \text{ is } \begin{vmatrix} d & f \\ g & i \end{vmatrix}.$$

$$\begin{vmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ d & e & \cancel{f} \\ g & h & \cancel{i} \end{vmatrix} \quad \text{The minor of } c \text{ is } \begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

To use expansion by minors with third-order determinants, each member of one row is multiplied by its minor and its *position sign*, and the results are added together. The position signs alternate between positive and negative, beginning with a positive sign in the first row, first column.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

KEY CONCEPT — *Third-Order Determinant*

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

The definition of third-order determinants shows an expansion using the elements in the first row of the determinant. However, any row can be used.

EXAMPLE Expansion by Minors

2 **Evaluate** $\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix}$ **using expansion by minors.**

Decide which row of elements to use for the expansion. For this example, we will use the first row.

$$\begin{vmatrix} 2 & 7 & -3 \\ -1 & 5 & -4 \\ 6 & 9 & 0 \end{vmatrix} = 2\begin{vmatrix} 5 & -4 \\ 9 & 0 \end{vmatrix} - 7\begin{vmatrix} -1 & -4 \\ 6 & 0 \end{vmatrix} + (-3)\begin{vmatrix} -1 & 5 \\ 6 & 9 \end{vmatrix} \quad \text{Expansion by minors}$$

$$= 2(0 - (-36)) - 7(0 - (-24)) - 3(-9 - 30) \quad \text{Evaluate determinants.}$$

$$= 2(36) - 7(24) - 3(-39)$$

$$= 72 - 168 + 117 \text{ or } 21 \quad \text{Multiply.}$$

CHECK Your Progress

2. Evaluate $\begin{vmatrix} -2 & 3 & -1 \\ 5 & -3 & 8 \\ 4 & -6 & -5 \end{vmatrix}$ using expansion by minors.

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Another method for evaluating a third-order determinant is by using diagonals.

Step 1 Begin by writing the first two columns on the right side of the determinant.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

Step 2 Next, draw diagonals from each element of the top row of the determinant downward to the right. Find the product of the elements on each diagonal.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

$aei \quad bfg \quad cdh$

Then, draw diagonals from the elements in the third row of the determinant upward to the right. Find the product of the elements on each diagonal.

$gec \quad hfa \quad idb$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix}$$

Step 3 To find the value of the determinant, add the products of the first set of diagonals and then subtract the products of the second set of diagonals. The sum is $aei + bfg + cdh - gec - hfa - idb$.

EXAMPLE Use Diagonals

3 **Evaluate** $\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}$ **using diagonals.**

Step 1 Rewrite the first two columns to the right of the determinant.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}\begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

Step 2 Find the products of the elements of the diagonals.

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}\begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

$4 \quad 0 \quad 60$

$0 \quad -5 \quad 24$

$$\begin{vmatrix} -1 & 3 & -3 \\ 4 & -2 & -1 \\ 0 & -5 & 2 \end{vmatrix}\begin{matrix} -1 & 3 \\ 4 & -2 \\ 0 & -5 \end{matrix}$$

Step 3 Add the bottom products and subtract the top products.
$4 + 0 + 60 - 0 - (-5) - 24 = 45$

The value of the determinant is 45.

CHECK Your Progress

3. Evaluate $\begin{vmatrix} 1 & -5 & 3 \\ 0 & 2 & -7 \\ 5 & -1 & -2 \end{vmatrix}$ using diagonals.

One very useful application of determinants is finding the areas of polygons. The formula below shows how determinants can be used to find the area of a triangle using the coordinates of the vertices.

Study Tip

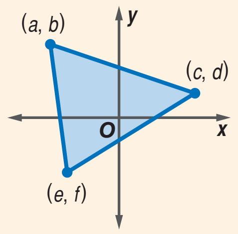
Area Formula

Notice that it is necessary to use the absolute value of A to guarantee a nonnegative value for the area.

KEY CONCEPT *Area of a Triangle*

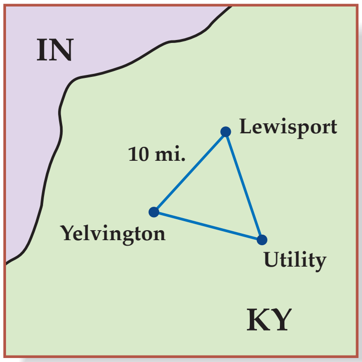
The area of a triangle having vertices at (a, b), (c, d), and (e, f) is $|A|$, where

$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}.$$



Real-World EXAMPLE

4 **RADIO** **A local radio station in Kentucky wants to place a tower that is strong enough to cover the cities of Yelvington, Utility, and Lewisport. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Kentucky with Yelvington at the origin, the coordinates of the three cities are (0, 0), (3, 0), and (1, 2). Use a determinant to estimate the area the signal must cover.**



$$A = \frac{1}{2}\begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$ Area Formula

$$= \frac{1}{2}\begin{vmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$$ $(a, b) = (3, 0)$, $(c, d) = (0, 2)$, $(e, f) = (0, 0)$

$$= \frac{1}{2}\left[3\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix}\right]$$ Expansion by minors

$$= \frac{1}{2}[3(2 - 0) - 0(1 - 0) + 1(0 - 0)]$$ Evaluate 2×2 determinants.

$$= \frac{1}{2}(6 - 0 - 0)$$ Multiply.

$$= \frac{1}{2}(6) \text{ or } 3$$ Simplify.

Remember that 1 unit equals 10 miles, so 1 square unit = 10×10 or 100 square miles. Thus, the area is 3×100 or 300 square miles.

CHECK Your Progress

4. Find the area of the triangle whose vertices are located at (2, 3), (−4, −3), and (1, −2).

CHECK Your Understanding

Example 1 (p. 194)

Find the value of each determinant.

1. $\begin{vmatrix} 7 & 8 \\ 3 & -2 \end{vmatrix}$

2. $\begin{vmatrix} -3 & -6 \\ 4 & 8 \end{vmatrix}$

Example 2 (p. 195)

Evaluate each determinant using expansion by minors.

3. $\begin{vmatrix} 0 & -4 & 0 \\ 3 & -2 & 5 \\ 2 & -1 & 1 \end{vmatrix}$

4. $\begin{vmatrix} 2 & 3 & 4 \\ 6 & 5 & 7 \\ 1 & 2 & 8 \end{vmatrix}$

Example 3 (p. 196)

Evaluate each determinant using diagonals.

5. $\begin{vmatrix} 1 & 6 & 4 \\ -2 & 3 & 1 \\ 1 & 6 & 4 \end{vmatrix}$

6. $\begin{vmatrix} -1 & 4 & 0 \\ 3 & -2 & -5 \\ -3 & -1 & 2 \end{vmatrix}$

Example 4 (p. 197)

7. **GEOMETRY** What is the area of $\triangle ABC$ with $A(5, 4)$, $B(3, -4)$, and $C(-3, -2)$?

8. Find the area of the triangle whose vertices are located at $(2, -1)$, $(1, 2)$, and $(-1, 0)$.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
9–16	1
17–22	2
23–25	3
26–29	4

Find the value of each determinant.

9. $\begin{vmatrix} 10 & 6 \\ 5 & 5 \end{vmatrix}$

10. $\begin{vmatrix} 8 & 5 \\ 6 & 1 \end{vmatrix}$

11. $\begin{vmatrix} -7 & 3 \\ -9 & 7 \end{vmatrix}$

12. $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix}$

13. $\begin{vmatrix} -6 & -2 \\ 8 & 5 \end{vmatrix}$

14. $\begin{vmatrix} -9 & 0 \\ -12 & -7 \end{vmatrix}$

15. $\begin{vmatrix} 7 & 5.2 \\ -4 & 1.6 \end{vmatrix}$

16. $\begin{vmatrix} -3.2 & -5.8 \\ 4.1 & 3.9 \end{vmatrix}$

17. $\begin{vmatrix} 3 & 1 & 2 \\ 0 & 6 & 4 \\ 2 & 5 & 1 \end{vmatrix}$

18. $\begin{vmatrix} 7 & 3 & -4 \\ -2 & 9 & 6 \\ 0 & 0 & 0 \end{vmatrix}$

19. $\begin{vmatrix} -2 & 7 & -2 \\ 4 & 5 & 2 \\ 1 & 0 & -1 \end{vmatrix}$

20. $\begin{vmatrix} -3 & 0 & 6 \\ 6 & 5 & -2 \\ 1 & 4 & 2 \end{vmatrix}$

21. $\begin{vmatrix} 1 & 5 & -4 \\ -7 & 3 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

22. $\begin{vmatrix} 3 & 7 & 6 \\ -1 & 6 & 2 \\ 8 & -3 & -5 \end{vmatrix}$

23. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 9 & 5 \\ 8 & 7 & 4 \end{vmatrix}$

24. $\begin{vmatrix} 1 & 5 & 2 \\ -6 & -7 & 8 \\ 5 & 9 & -3 \end{vmatrix}$

25. $\begin{vmatrix} 8 & -9 & 0 \\ 1 & 5 & 4 \\ 6 & -2 & 3 \end{vmatrix}$

26. **GEOGRAPHY** Mr. Cardona is a regional sales manager for a company in Florida. Tampa, Orlando, and Ocala outline his region. If a coordinate grid in which 1 unit = 10 miles is placed over the map of Florida with Tampa at the origin, the coordinates of the three cities are $(0, 0)$, $(7, 5)$, and $(2.5, 10)$. Estimate the area of his sales territory.

27. **ARCHAEOLOGY** During an archaeological dig, a coordinate grid is laid over the site to identify the location of artifacts as they are excavated. Suppose three corners of a building have been unearthed at $(-1, 6)$, $(4, 5)$, and $(-1, -2)$. If each square on the grid measures one square foot, estimate the area of the floor of the building, assuming that it is triangular.

Real-World Career

Archaeologist

Archaeologists attempt to reconstruct past ways of life by examining preserved bones, the ruins of buildings, and artifacts such as tools, pottery, and jewelry.

Math Online

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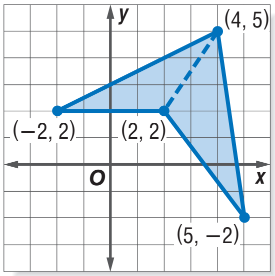
28. **GEOMETRY** Find the area of a triangle whose vertices are located at $(4, 1)$, $(2, -1)$, and $(0, 2)$.

29. **GEOMETRY** Find the area of the polygon shown at the right.



30. Solve for x if $\det\begin{bmatrix} 2 & x \\ 5 & -3 \end{bmatrix} = 24$.

31. Solve $\det\begin{bmatrix} 4 & x & -2 \\ -x & -3 & 1 \\ -6 & 2 & 3 \end{bmatrix} = -3$ for x.

32. **GEOMETRY** Find the value of x such that the area of a triangle whose vertices have coordinates $(6, 5)$, $(8, 2)$, and $(x, 11)$ is 15 square units.

33. **GEOMETRY** The area of a triangle ABC is 2 square units. The vertices of the triangle are $A(-1, 5)$, $B(3, 1)$, and $C(-1, y)$. What are the possible values of y?

MATRIX FUNCTION You can use a TI-83/84 Plus to find determinants of square matrices using the **MATRIX** functions. Enter the matrix under the **EDIT** menu. Then from the home screen choose **det(**, which is option 1 on the **MATH** menu, followed by the matrix name to calculate the determinant.

Use a graphing calculator to find the value of each determinant.

34. $\begin{bmatrix} 3 & -6.5 \\ 8 & 3.75 \end{bmatrix}$

35. $\begin{bmatrix} 10 & 20 & 30 \\ 40 & 50 & 60 \\ 70 & 80 & 90 \end{bmatrix}$

36. $\begin{bmatrix} 10 & 12 & 4 \\ -3 & 18 & -9 \\ 16 & -2 & -1 \end{bmatrix}$

H.O.T. Problems

37. **OPEN ENDED** Write a matrix whose determinant is zero.

38. **FIND THE ERROR** Khalid and Erica are finding the determinant of $\begin{bmatrix} 8 & 3 \\ -5 & 2 \end{bmatrix}$. Who is correct? Explain your reasoning.

Khalid

$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - (-15)$

$= 31$

Erica

$\begin{vmatrix} 8 & 3 \\ -5 & 2 \end{vmatrix} = 16 - 15$

$= 1$

EXTRA PRACTICE

See pages 898, 929.

Math Online

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39. **REASONING** Find a counterexample to disprove the following statement.

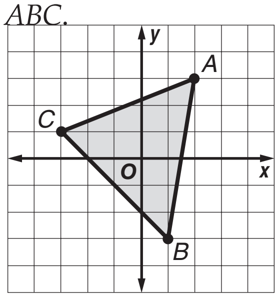
Two different matrices can never have the same determinant.

40. **CHALLENGE** Find a third-order determinant in which no element is 0, but for which the determinant is 0.

41. ***Writing in Math*** Use the information about the "Bermuda Triangle" on page 194 to explain how matrices can be used to find the area covered in this triangle. Then use your method to find the area.

STANDARDIZED TEST PRACTICE

42. ACT/SAT Find the area of triangle *ABC*.



A 10 units^2

B 12 units^2

C 14 units^2

D 16 units^2

43. REVIEW Use the table to determine the expression that best represents the number of faces of any prism having a base with n sides.

Base	Sides of Base	Faces of Prisms
Triangle	3	5
Quadrilateral	4	6
Pentagon	5	7
Hexagon	6	8
Heptagon	7	9
Octagon	8	10

F $2(n - 1)$ **H** $n + 2$

G $2(n + 1)$ **J** $2n$

Spiral Review

For Exercises 44 and 45, use the following information. (Lesson 4-4)
The vertices of $\triangle ABC$ are $A(-2, 1)$, $B(1, 2)$ and $C(2, -3)$. The triangle is dilated so that its perimeter is $2\frac{1}{2}$ times the original perimeter.

44. Write the coordinates of $\triangle ABC$ in a vertex matrix.

45. Find the coordinates of $\triangle A'B'C'$. Then graph $\triangle ABC$ and $\triangle A'B'C'$.

Find each product, if possible. (Lesson 4-3)

46. $\begin{bmatrix} 2 & 4 \\ -2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 3 & 9 \\ -1 & 2 \end{bmatrix}$

47. $\begin{bmatrix} 5 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ -4 & 2 \end{bmatrix}$

48. $\begin{bmatrix} 7 & -5 & 4 \\ 6 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -8 \\ 1 & 2 \end{bmatrix}$

49. MARATHONS The length of a marathon was determined in the 1908 Olympic Games in London, England. The race began at Windsor Castle and ended in front of the royal box at London's Olympic Stadium, which was a distance of 26 miles 385 yards. Determine how many feet the marathon covers using the formula $f(m, y) = 5280m + 3y$, where m is the number of miles and y is the number of yards. (Lesson 3-4)

Write an equation in slope-intercept form for the line that satisfies each set of conditions. (Lesson 2-4)

50. slope 1, passes through (5, 3)

51. slope $-\frac{4}{3}$, passes through (6, –8)

52. passes through (3, 7) and (–2, –3)

53. passes through (0, 5) and (10, 10)

GET READY for the Next Lesson

PREREQUISITE SKILL Solve each system of equations. (Lesson 3-2)

54. $x + y = -3$
$3x + 4y = -12$

55. $x + y = 10$
$2x + y = 11$

56. $2x + y = 5$
$4x + y = 9$